UNPUBLISHED PRELIMINARY DATA

MH MPG Report 1541-TR 10

12 December 1963

7/p

N64 14289

Code 1

CR-55351

Honeywell

ROCKET BOOSTER CONTROL

SECTION 10

A CONCISE FORMULATION OF A BOUNDED PHASE COORDINATE CONTROL PROBLEM AS A PROBLEM IN THE CALCULUS OF VARIATIONS

NASA Contract NASw-563

OTS PRICE

XEROX $ 1.10 ph

MICROFILM $ 0.80 mf.

MILITARY PRODUCTS GROUP RESEARCH DEPARTMENT

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NASA Contract NASw-563

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FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

1541-TR 1	Summary
1541-TR 2	Control of Plants Whose Representation Contains Derivatives of the Control Variable
1541-TR 3	Modes of Finite Response Time Control
1541-TR 4	A Sufficient Condition in Optimal Control
1541-TR 5	Time Optimal Control of Linear Recurrence Systems
1541-TR 6	Time-Optimal Bounded Phase Coordinate Control of Linear Recurrence Systems
1541-TR 7	Penalty Functions and Bounded Phase Coordinate Control
1541-TR 8	Linear Programming and Bounded Phase Coordinate Control
1541-TR 9	Time Optimal Control with Amplitude and Rate Limited Controls
1541-TR 10	A Concise Formulation of a Bounded Phase Coordinate Control Problem as a Problem in the Calculus of Variations
1541-TR 11	A Note on System Truncation
1541-TR 12	State Determination for a Flexible Vehicle Without a Mode Shape Requirement
1541-TR 13	An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle
1541-TR 14	Minimum Disturbance Effects Control of Linear Systems with Linear Controllers
1541-TR 15	An Alternate Derivation and Interpretation of the Drift-Minimum Principle
1541-TR 16	A Minimax Control for a Plant Subjected to a Known Load Disturbance

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components ($1 < m \leq n$), of the state vector for an n-th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

A CONCISE FORMULATION OF A BOUNDED PHASE COORDINATE CONTROL PROBLEM AS A PROBLEM IN THE CALCULUS OF VARIATIONS*

By W. Schmaedeke+ and D. Russell‡

ABSTRACT

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A short calculation is presented which transforms a bounded phase coordinate control problem for a linear time varying system into a problem in the calculus of variations.

AUTHOR

THE CALCULATION

The linear system is given by

$$\dot{x} = A(t)x + B(t)u + c(t) \tag{1}$$

where A(t) is an nxn matrix, B(t) is an nxr matrix, c(t) is an n-vector, x is an n-vector and u is an m-vector. The constraints on the amplitude of the control vector are given by

$$a_{1i}(t) \leq u_i(t) \leq a_{2i}(t);\ i=1,\ldots,m \tag{2}$$

and the constraints on the phase variables are given by

$$v_r(t)\cdot x(t) \leq b_r(t) \quad r = 1,\ldots,R \tag{3}$$

where $v_r(t)$ are given bounded, vector functions of time and $b_r(t)$ are given bounded scalar functions defined on an interval I; $a_{1i}(t)$, $a_{2i}(t)$ are bounded and also defined on I. The coefficients A(t), B(t) and c are assumed to have elements which are bounded and continuous on the interval I.

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It can be shown that the problem of transferring a trajectory from an initial point at time $t = 0$ to the origin in a minimum time T can be phrased as the problem of choosing $u(t)$ from some given admissible set Ω of controls ($\Omega \subset R^m$) in such a way that the quantity

$$\int_0^T \psi'(t)B(t)\, u(t)dt \tag{4}$$

is maximized subject to the constraints given by (2) and (3) with $\psi(t)$ a solution to the adjoint equations corresponding to (1).

The variation of parameters formula for a solution $x(t)$ of (1) is

$$x(t) = W(t)\, x_o + W(t) \int_0^t W^{-1}(s)B(s)u(s)ds \tag{5}$$

where $W(t)$ is a fundamental solution matrix to the homogeneous part of (1). By substituting (5) into (3)

$$v_r \cdot x = v_r \cdot W(t)\, x_o + v_r \cdot W(t) \int_0^t W^{-1}(s)B(s)u(s)ds \leq b_r(t)$$

or since $v_r \cdot W(t)x_o$ is some scalar function of time,

$$v_r(t) \cdot W(t) \int_0^t W^{-1}(s)B(s)u(s)ds \leq \tilde{b}_r(t) \tag{6}$$

where $v_r(t) \cdot W(t)x_o$ has been absorbed in the scalar function $\tilde{b}_r(t)$.

$I = W(s)W^{-1}(s)$ is introduced into equation 4 as follows:

$$\int_0^t \psi'(s)W(s)W^{-1}(s)B(s)u(s)ds. \tag{7}$$

It is observed that $\psi'(s)W(s)$ is a constant vector which is denoted by K, and hence the problem now is to maximize the quantity

$$K \cdot \int_0^T W^{-1}(s)B(s)u(s)ds \tag{8}$$

subject to the constraints

$$v_r(t) \cdot W(t) \int_0^t W^{-1}(s)B(s)u(s)ds \leq \tilde{b}_r(t) \tag{9}$$

Remark: This can be posed in terms of maximizing the projection of a certain vector onto the adjoint vector at the final time T by noting K may be replaced by $\psi(T)W(T)$ in (8) to obtain

$$\psi(T) \cdot \int_0^T W(T)W^{-1}(s)B(s)u(s)ds \tag{10}$$

while (9) becomes

$$v_r(t) \cdot W(t)W^{-1}(T) \int_0^T W(T)W^{-1}(s)B(s)u(s)ds \leq \tilde{b}_r(t) \tag{11}$$

These equations are of the form:

$$\text{maximize } \psi(T) \cdot \int_0^T F(s)u(s)ds \tag{12}$$

subject to the constraints

$$\alpha_r(t) \cdot \int_0^t F(s)u(s)ds \leq \beta_r(t) \tag{13}$$

and it is this final succinct form which the bounded phase problem takes.